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Stability of weakly nonlinear deep-water waves in two and three dimensions

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The stability of a weakly nonlinear wave train on deep water to two- and three-dimensional modulations is investigated using an improved approximation due to Zakharov (1968). The results are expressible in simple analytical forms, and show good quantitative agreement with available experimental data and exact numerical calculations over a broad range of wave steepness in the unidirectional case.

1. Introduction

Benjamin & Feir (1967) made the important discovery that a uniform train of weakly nonlinear deep-water waves is unstable to long-wave perturbations. Zakharov (1968) showed that modulations of weakly nonlinear deep-water waves are describable by a nonlinear Schrödinger equation, and the instability results of Benjamin & Feir can be recovered. In addition, the nonlinear Schrödinger equation can be solved exactly for certain initial conditions to yield stable envelope solitons (Zakharov & Shabat 1971).

The instability of a uniform wave train to three-dimensional perturbations‡ has been determined by Zakharov (1968) using the nonlinear Schrödinger equation. The instability of other steady plane envelope solutions, namely the envelope soliton and the cnoidal solutions, to three-dimensional perturbations has been established by Zakharov & Rubenchik (1973), Saffman & Yuen (1978) and Martin, Saffman & Yuen (1980).

The nonlinear Schrödinger equation can also be used to study the long-time evolution of the unstable wave train. In two dimensions, Lake *et al.* (1977) identified the phenomenon of recurrence, which was further studied by Yuen & Ferguson (1978*a*). The long-time evolution of an unstable wave train in three dimensions has been investigated by Yuen & Ferguson (1978*b*) and Martin & Yuen (1980).

Despite the impressive agreement between the theoretical predictions based on the nonlinear Schrödinger equation and laboratory experimental data (Lake *et al.* 1977), the equation appears deficient in two respects which suggests the need for a higher-order approximation. First, the quantitative agreement demonstrated by Lake *et al.* (1977) relied on a questionable argument (see § 3.1 below). Second, the instability

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‡ We use the term 'three-dimensional' to signify the existence of variations in the crest-wise direction, the term 'two-dimensional' refers to plane disturbances. This set of notations acknowledges the depth dependence of the motion of water particles in deep-water wave problems.

diagram for three-dimensional disturbances on a uniform wave train does not possess high-wavenumber cutoff. Consequently, as demonstrated by Martin & Yuen (1980), the equation allows the leakage of energy to higher modes outside its range of validity and can become inconsistent (i.e. not uniformly valid for long times).

In this paper, we use a more accurate equation given by Zakharov (1968) (from which the nonlinear Schrödinger equation was derived with further approximations) to re-investigate the stability properties of uniform deep-water wave trains to two- and three-dimensional modulations. We shall demonstrate that the results from Zakharov's equation agree quantitatively in the common ranges of application with the numerical calculations for the full water wave equations by Longuet-Higgins (1978) and the analytical result derivable from Whitham's theory (Peregrine & Thomas 1979). The results of Longuet-Higgins (1978) are for two-dimensional modulations, and those of Peregrine & Thomas (1979) are confined to the limit of infinitely long perturbations. Our results give an improved stability diagram for three-dimensional perturbations which exhibits high-wavenumber cutoffs and re-stabilization for large wave steepness in both two and three dimensions. Furthermore, comparison with experimental data of Lake *et al.* (1977) and Benjamin (1967) yields excellent quantitative agreement without reliance on any extrinsic arguments.

It thus appears that Zakharov's equation provides an improved description of the dynamics of nonlinear water waves. In fact, it has been used in the study of three-dimensional bifurcation of steady waves (Saffman & Yuen 1980*a*, *b*) and properties of random inhomogeneous waves (Crawford, Saffman & Yuen 1980) with interesting results.

2. The governing equation

It has been shown by Zakharov (1968) that the governing equation for weakly nonlinear deep-water waves, correct to third order in amplitude, is

$$i\frac{\partial B(\mathbf{k},t)}{\partial t} = \iiint_{-\infty}^{\infty} T(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\,\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\exp\{i[\omega(\mathbf{k})+\omega(\mathbf{k}_1) - \omega(\mathbf{k}_2)-\omega(\mathbf{k}_3)]t\}\,B^*(\mathbf{k}_1,t)\,B(\mathbf{k}_2,t)\,B(\mathbf{k}_3,t)\,d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3, \quad (1)$$

where $B(\mathbf{k},t)$ can be interpreted as the spectral component of the wave envelope, being related to the free surface $\eta(\mathbf{x},t)$ by the expression

$$\eta(\mathbf{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{|\mathbf{k}|}{2\omega(\mathbf{k})}\right)^{\frac{1}{2}}\{B(\mathbf{k},t)\exp\{i[\mathbf{k}.\mathbf{x}-\omega(\mathbf{k})\,t]\} + B^*(\mathbf{k},t)\exp\{-i[\mathbf{k}.\mathbf{x}-\omega(\mathbf{k})\,t]\}\}\,d\mathbf{k}, \quad (2)$$

$(\ )^*$ denotes the complex conjugate, $\mathbf{k} = (k,l)$ is the wave vector, $\mathbf{x} = (x,y)$ is the horizontal spatial vector, and ω is the linearized wave frequency in radians per second, which is related to $\mathbf{k}$ through the dispersion relation and which is given by $\omega(\mathbf{k}) = (g|\mathbf{k}|)^{\frac{1}{2}}$ with g as the acceleration due to gravity. $T(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$ is a real scalar coupling coefficient given in the appendix.

We consider $B(\mathbf{k},t)$ as a superposition of discrete modes

$$B(\mathbf{k},t) = \sum_n B_n(t)\,\delta(\mathbf{k}-\mathbf{k}_n). \quad (3)$$

Substituting (3) into (1) and evaluating the delta functions, we obtain for each mode k_p the discrete equation

$$i\frac{dB_p}{dt} = \sum_i \sum_j \sum_m \delta_{p+i-j-m} T_{p,i,j,m} \exp\{i(\omega_p+\omega_i-\omega_j-\omega_m)t\} B_i^* B_j B_m, \tag{4}$$

where $\omega_i = \omega(\mathbf{k}_i)$, the generalized Kronecker delta $\delta_{p+i-j-m}$ denotes that summation is taken over those subscripts satisfying

$$\mathbf{k}_p + \mathbf{k}_i = \mathbf{k}_j + \mathbf{k}_m, \tag{5}$$

and $T_{p,i,j,m}$ denotes $T(\mathbf{k}_p, \mathbf{k}_i, \mathbf{k}_j, \mathbf{k}_m)$. At this stage we note that if we specialize to those modes for which

$$\omega_p + \omega_i = \omega_j + \omega_m \tag{6}$$

in addition to the constraint (5), we would arrive at the discrete resonant interaction equation first derived by Benney (1962):

$$i\frac{dB_p}{dt} = \sum_i \sum_j \sum_m \delta_{p+i-j-m} T_{p,i,j,m} B_i^* B_j B_m. \tag{7}$$

3. Stability of a uniform wave train

The solution of equation (4) corresponding to a uniform wave train of wave vector $\mathbf{k}_0 = (k_0, 0)$ is

$$\begin{aligned} B_p(t) &= B_0 \exp(-iT_0 B_0^2 t) \quad \text{for} \quad \mathbf{k}_p = \mathbf{k}_0, \\ &= 0 \qquad\qquad\qquad\quad \text{for} \quad \mathbf{k}_p \neq \mathbf{k}_0, \end{aligned} \tag{8}$$

where $T_0 = T(\mathbf{k}_0, \mathbf{k}_0, \mathbf{k}_0, \mathbf{k}_0) = k_0^3/4\pi^2$, and $B_0 = \pi a_0 (2\omega_0/k_0)^{\frac{1}{2}}$. The amplitude of the carrier wave is a_0. We impose perturbations represented by a pair of wave vectors $\mathbf{k}_0 \pm \mathbf{K}$ with amplitudes $B_\pm(t)$. Neglecting the squares of small quantities, it follows from equation (4) that $B_\pm(t)$ satisfy

$$i\frac{dB_\pm}{dt} = T_{\pm,\mp} B_0^2 B_\mp^* \exp[-i(\Omega + 2T_0 B_0^2)t] + 2T_{\pm,\pm} B_0^2 B_\pm, \tag{9}$$

where we have introduced the simplified notation

$$\begin{aligned} T_{\pm,\pm} &= T(\mathbf{k}_0 \pm \mathbf{K}, \mathbf{k}_0, \mathbf{k}_0, \mathbf{k}_0 \pm \mathbf{K}), \\ T_{\pm,\mp} &= T(\mathbf{k}_0 \pm \mathbf{K}, \mathbf{k}_0 \mp \mathbf{K}, \mathbf{k}_0, \mathbf{k}_0), \\ \Omega &= 2\omega(\mathbf{k}_0) - \omega(\mathbf{k}_0 + \mathbf{K}) - \omega(\mathbf{k}_0 - \mathbf{K}). \end{aligned} \tag{10}$$

Substituting

$$\begin{aligned} B_+ &= \hat{B}_+ \exp[-i(\tfrac{1}{2}\Omega + T_0 B_0^2)t - i\sigma t], \\ B_-^* &= \hat{B}_-^* \exp[i(\tfrac{1}{2}\Omega + T_0 B_0^2)t - i\sigma t], \end{aligned} \tag{11}$$

we obtain a second-order eigenvalue problem in σ, $\hat{B}_+$ and $\hat{B}_-^*$ which has no explicit time dependence. The eigenvalues σ are the roots of

$$\begin{aligned} \sigma^2 + 2B_0^2(T_{-,-} - T_{+,+})\sigma + T_{+,-}T_{-,+}B_0^4 - (-\tfrac{1}{2}\Omega - T_0 B_0^2 + 2T_{-,-}B_0^2) \\ \times(-\tfrac{1}{2}\Omega - T_0 B_0^2 + 2T_{+,+}B_0^2) = 0. \end{aligned} \tag{12}$$

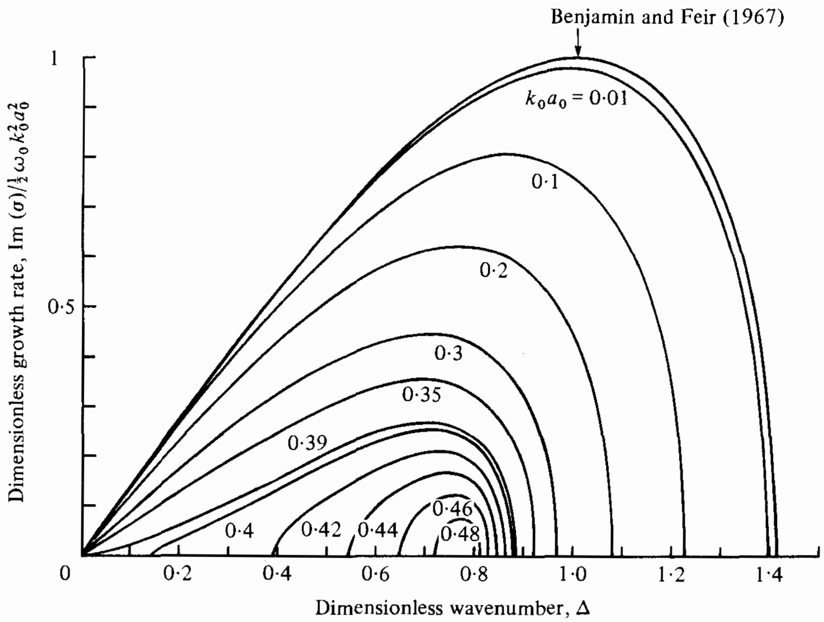


FIGURE 1. Instability growth rate as a function of perturbation wavenumber for various values of wave steepness.

The solutions are

$$\sigma = (T_{+,+} - T_{-,-}) B_0^2 \pm \{-T_{+,-} T_{-,+} B_0^4 + [-\tfrac{1}{2}\Omega + B_0^2(T_{+,+} + T_{-,-} - T_0)]^2\}^{\frac{1}{2}}. \quad (13)$$

The expression for σ given by (13) gives $\operatorname{Im}\sigma$ correct to $O(k_0^2 a_0^2)$, with no requirement that $|\mathbf{K}|$ be small compared with $|\mathbf{k}_0|$. However, the original approximations contained in the derivation of equation (4) require that Ω be of the order of $T_0 B_0^2$ for the mechanisms described to be dominant.

3.1. *Case of two-dimensional modulation*

In this case, we put $\mathbf{K} = (K_x, 0)$ and define the dimensionless perturbation wavenumber $\kappa = K_x/k_0$. From the dispersion relation, it can be seen that, for Ω to be small, κ must be small. In other words, the perturbations that possess the growth rate given by (13) must be 'sidebands' representing long-wave modulations. In the limit of very long perturbational wavelength, i.e. $\kappa \ll 1$, we can expand the dispersion relation in powers of κ:

$$\omega_{\pm} = \omega_0 \pm \tfrac{1}{2}\omega_0 \kappa - \tfrac{1}{8}\omega_0 \kappa^2. \quad (14)$$

Therefore

$$\Omega = 2\omega_0 - \omega_+ - \omega_- = \tfrac{1}{4}\omega_0 \kappa^2. \quad (15)$$

Expanding the expressions for $T_{\pm,\pm}$, $T_{\pm,\mp}$ in powers of κ and retaining only terms to $O(k_0^2 a_0^2)$ and $O(\kappa^2)$ in equation (13), we obtain

$$\sigma = \omega_0 \left(-\frac{\kappa^2}{8} k_0^2 a_0^2 + \frac{\kappa^4}{64}\right)^{\frac{1}{2}}, \quad (16)$$

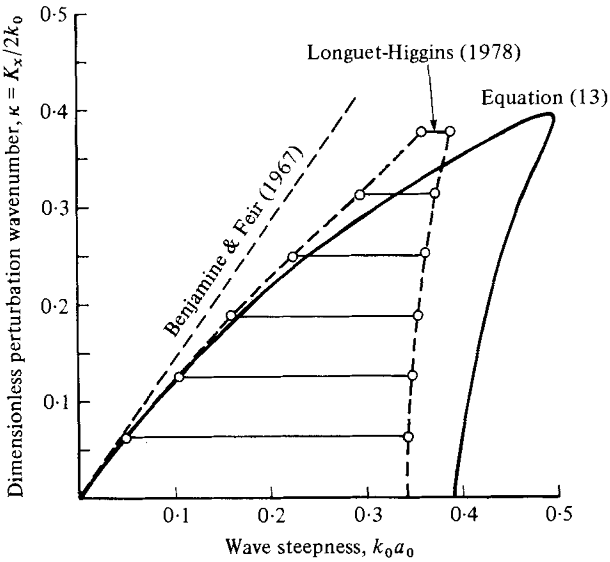


FIGURE 2. Stability boundary for growth of unstable perturbations for two-dimensional wave trains (comparison with results of Longuet-Higgins [1978]).

since $T_0B_0^2 = \frac{1}{2}k_0^2a_0^2\omega_0$. This result is identical with the instability first discussed by Benjamin & Feir (1967), and to that which follows from a stability analysis of the nonlinear Schrödinger equation (Hasimoto & Ono 1972; Lake *et al.* 1977).

We shall now show that equation (16) is not a very good approximation of equation (13) for moderate but small values of k_0a_0. To illustrate this, we show in figure 1 a plot of the instability growth rate $\mathrm{Im}\,\sigma$ [as obtained from equation (13)] as a function of the normalized perturbation wavenumber

$$\Delta = \kappa/2k_0a_0 \tag{17}$$

for various values of k_0a_0. The result of equation (16) [which is that of Benjamin & Feir (1967) as labelled] is approached when $k_0a_0 \rightarrow 0$. For non-zero k_0a_0, departures arise for larger values of Δ. These departures become significant with increasing k_0a_0. For $k_0a_0 = 0{\cdot}2$, the prediction regarding the most unstable wavenumber and the maximum growth rate achieve disagreement as large as 30 %. Furthermore, equation (13) predicts that the very long waves begin to become stable for k_0a_0 about 0·39. This restabilization of the very long waves agrees qualitatively with the results of Whitham's theory which yield long-wave restabilization at $k_0a_0 = 0{\cdot}34$. The quantitative discrepancy of 14 % is better than expected, since the present theory is formally accurate only to $O(k_0^2a_0^2)$.

Figure 1 also shows the trend towards restabilization of the entire system for sufficiently large k_0a_0 (at about $k_0a_0 = 0{\cdot}50$). This feature qualitatively agrees with the numerical results obtained from exact water wave equations by Longuet-Higgins (1978). A better illustration of this phenomenon is given in figure 2, where we have plotted the stability diagram in the (κ, k_0a_0) space. The numerical results of Longuet-Higgins (1978), confined to only discrete values of κ, are also plotted. It can be seen that the qualitative agreement is satisfactory overall, and quantitative agreement is

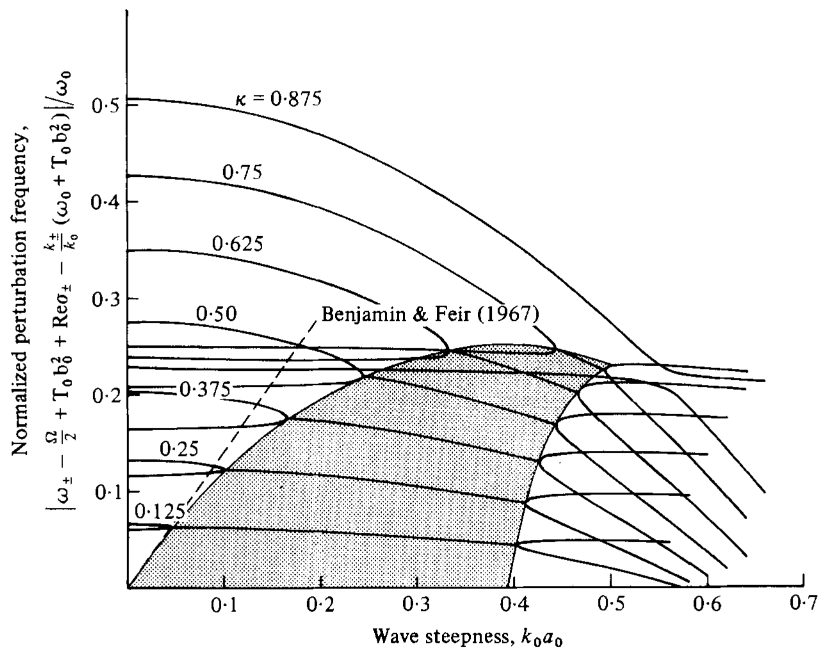


FIGURE 3. Normalized frequencies of the perturbation sidebands as a function of the wave steepness and perturbation wavenumber, $\kappa = K_x/k_0$. The co-ordinate system is fixed relative to the carrier wave. The shaded region indicates that σ is complex (unstable).

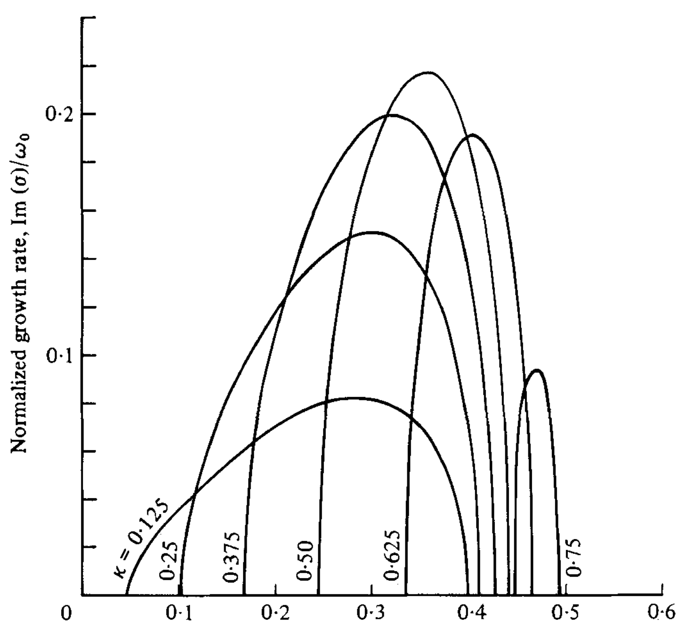


FIGURE 4. Instability growth rate as a function of wave steepness with perturbation wavenumber as a parameter.

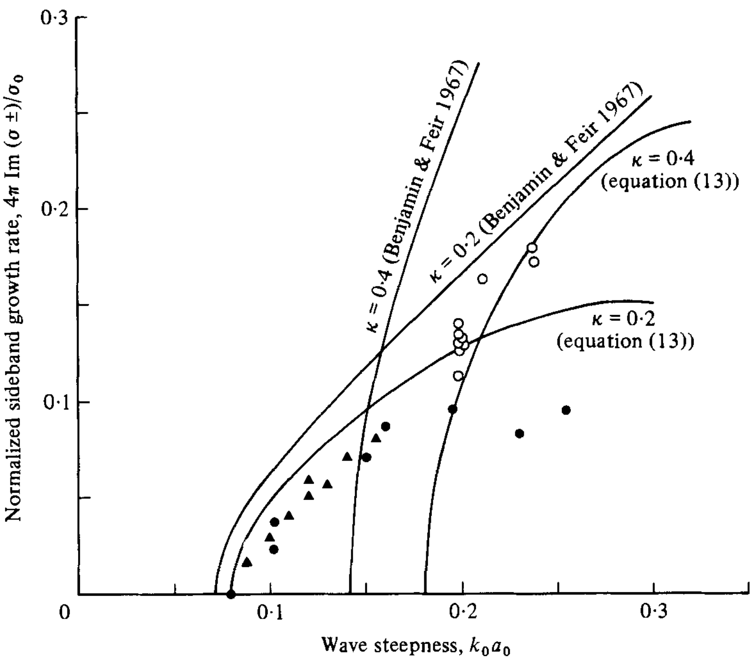


FIGURE 5. Comparison of calculated amplification rate with experimental results as a function of wave steepness. Experimental results: ○, $\kappa = 0·4$, Lake *et al.* (1977); ●, $\kappa = 0·2$, Lake *et al.* (1977); ▲, Benjamin (1967).

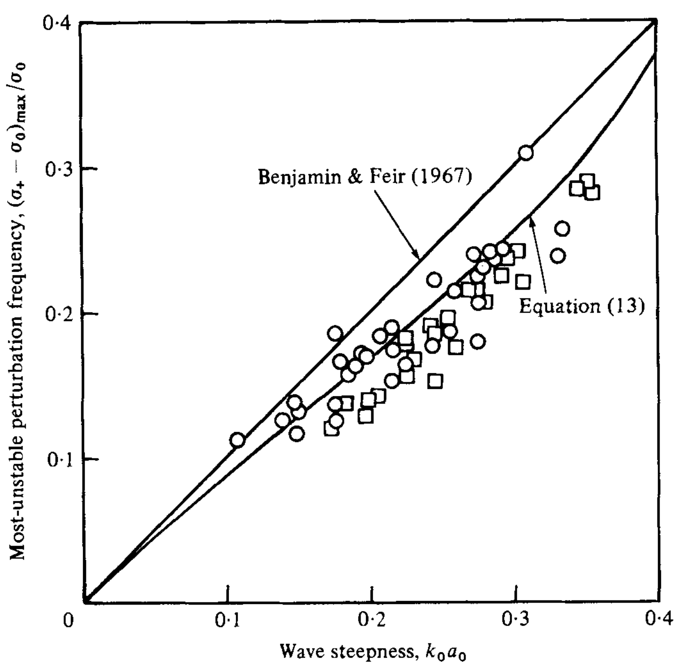


FIGURE 6. Maximum unstable frequency as a function of wave steepness. Data taken from figure 1 [with abscissa corresponding to scale labelled $(ka)_{5n}$] of Lake & Yuen (1977).

achieved for small and moderate values of $k_0 a_0$. In figures 3 and 4, we plot the real and imaginary parts of the perturbation frequency in the frame of reference moving with the individual wave crests. These plots can be compared with figures 5 and 6 of Longuet-Higgins (1978).† Note that no restabilization was predicted by the analysis of Benjamin & Feir (1967) or that based on the nonlinear Schrödinger equation.

For given values of κ (0·2 and 0·4), the predicted growth rate as a function of wave steepness has been compared to experimental data of Benjamin (1967) and Lake *et al.* (1977). The results are shown in figure 5. It can be seen that the agreement is quantitatively satisfactory. The Benjamin–Feir result is shown for comparison. In figure 6, we compare the predicted results on the most unstable perturbation frequency as a function of wave steepness with experimental data. Again, the agreement between theory and experiment is very good, whereas the result of Benjamin & Feir over-

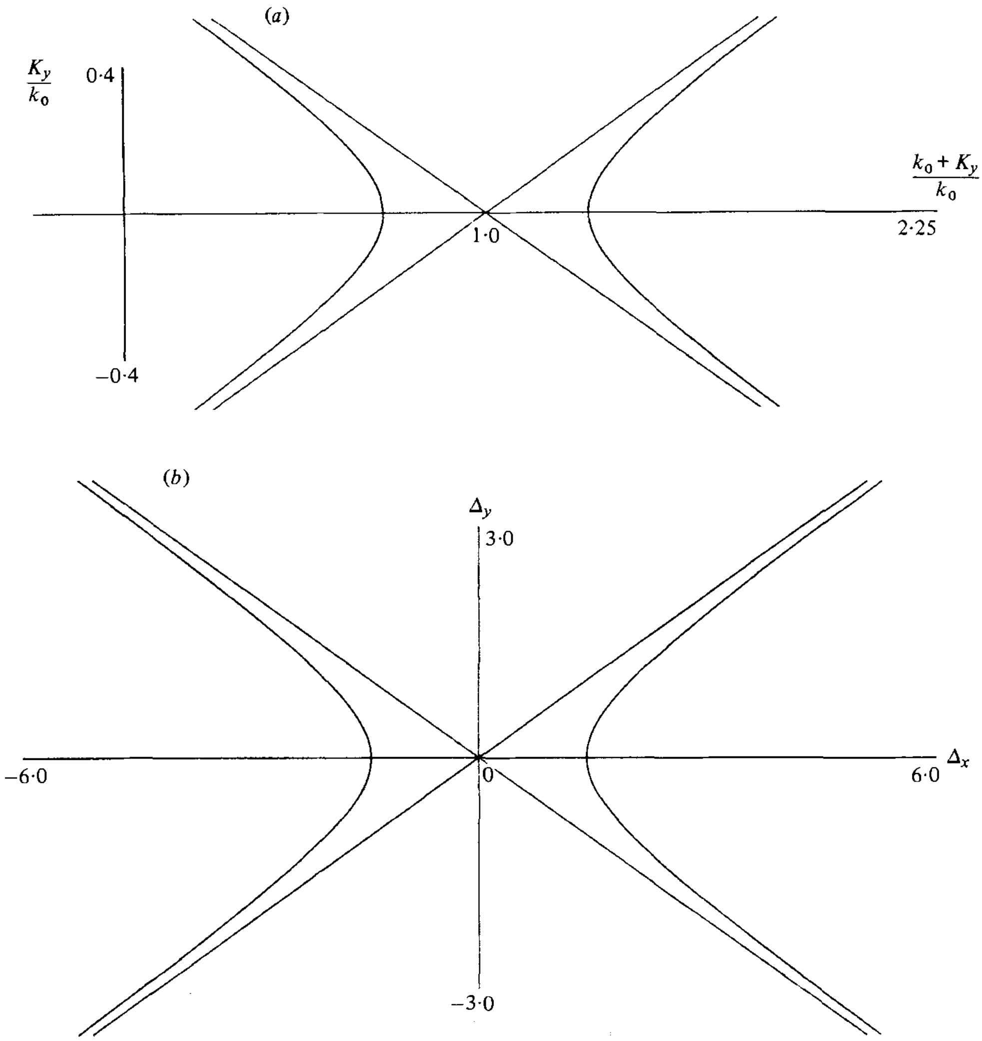
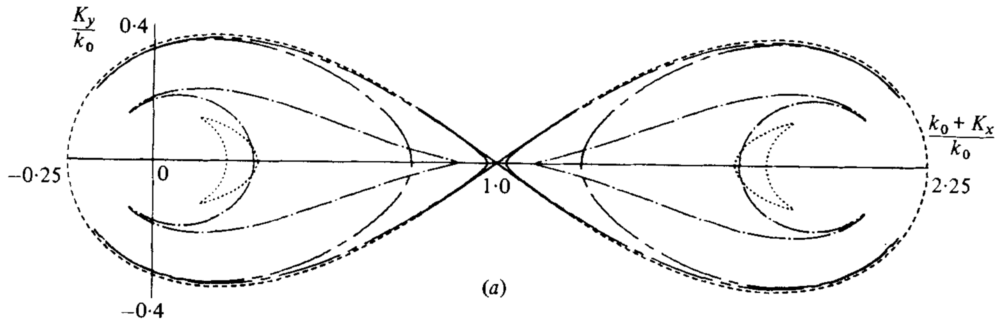


FIGURE 7. (a) Stability diagram in (K_x, K_y) space for the three-dimensional nonlinear Schrödinger equation. $k_0 a_0 = 0{\cdot}1$. (b) Stability diagram in (Δ_x, Δ_y) space, where $\Delta_x = K_x/2k_0^2 a_0$, $\Delta_y = K_y/2k_0^2 a_0$ for the three-dimensional nonlinear Schrödinger equation. The dependence on $k_0 a_0$ is completely scaled out in this co-ordinate system.

unstable regions predicted by equations (21) and (13) is that the latter is now finite in extent. For small values of $k_0 a_0$, the instability region lies adjacent to Phillips 'figure-of-eight' diagram ($\Omega \equiv 0$) which is valid for weakly nonlinear point spectra. As $k_0 a_0$ increases, the wave vectors near the edges of the 'figure-of-eight' stabilize, and the diagram approaches that of a pair of touching 'horseshoes'. For sufficiently large $k_0 a_0$, the longer waves also begin to stabilize, and the two 'horseshoes' split. Just before the total system stabilizes, the instability is concentrated at $\mathbf{K} \doteq \pm 0{\cdot}78\mathbf{k}_0$, and is strongly one-dimensional.

The fact that the instability region is finite in extent, and not infinite as predicted by the three-dimensional nonlinear Schrödinger equation, may have significant consequences in light of the findings of Yuen & Ferguson (1978*a*) and Martin & Yuen



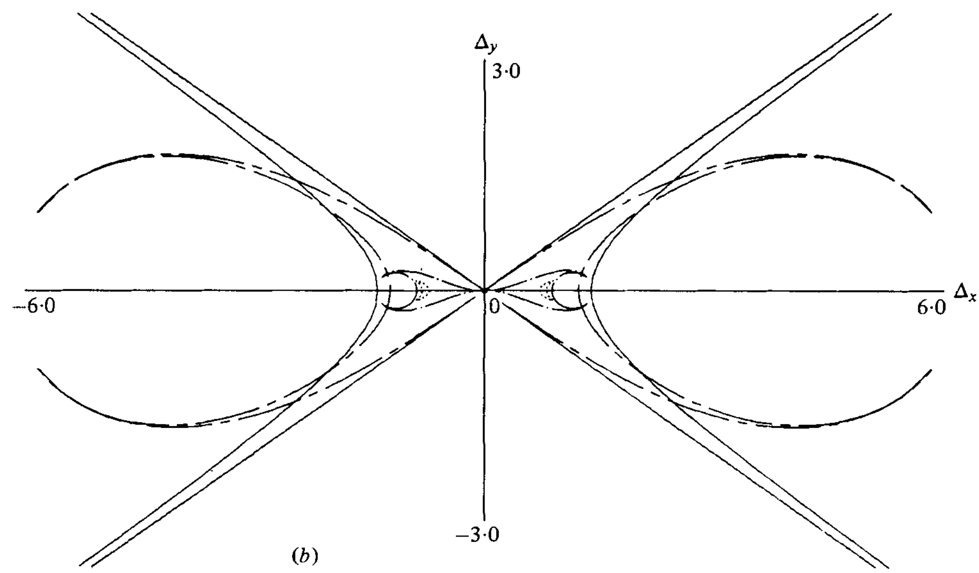


FIGURE 8. (a) Stability diagram in (K_x, K_y) space for various values of wave steepness for equation (13); – – –, $k_0 a_0 = 0$ (Phillips' figure-of-eight). (b) Stability diagram in (Δ_x, Δ_y) space, where $\Delta_x = K_x/2k_0^2 a_0$, $\Delta_y = K_y/2k_0^2 a_0$ for various values of wave steepness for equation (13). ——, $k_0 a_0 = 0{\cdot}01$; – – –, $k_0 a_0 = 0{\cdot}1$; – · –, $k_0 a_0 = 0{\cdot}4$; · · · ·, $k_0 a_0 = 0{\cdot}48$.

(1980). Yuen & Ferguson (1978a) performed numerical experiments on the time evolution of an unstable wave train using the two-dimensional nonlinear Schrödinger equation and demonstrated that only the prescribed unstable modes, and all their higher harmonics which are also unstable, actively participate in the evolutionary process. Since the instability extent is finite in the wavenumber range, given any unstable perturbation wavenumbers, only a finite number of modes dominate the evolution, and recurrence is highly likely. On the other hand, the instability region for three-dimensional modulation is infinite in extent according to the three-dimensional equation. It is expected that energy contained in the unstable modes can eventually leak out to higher and higher harmonics which are also unstable. This leakage of energy occurs in quasi-recurring fashion as found by Martin & Yuen (1980). Now that the instability region is found to be finite in extent even in three dimensions, we once again expect the evolution to be dominated by a finite number of unstable wave components, and that the leakage identified by Martin & Yuen should not exist in

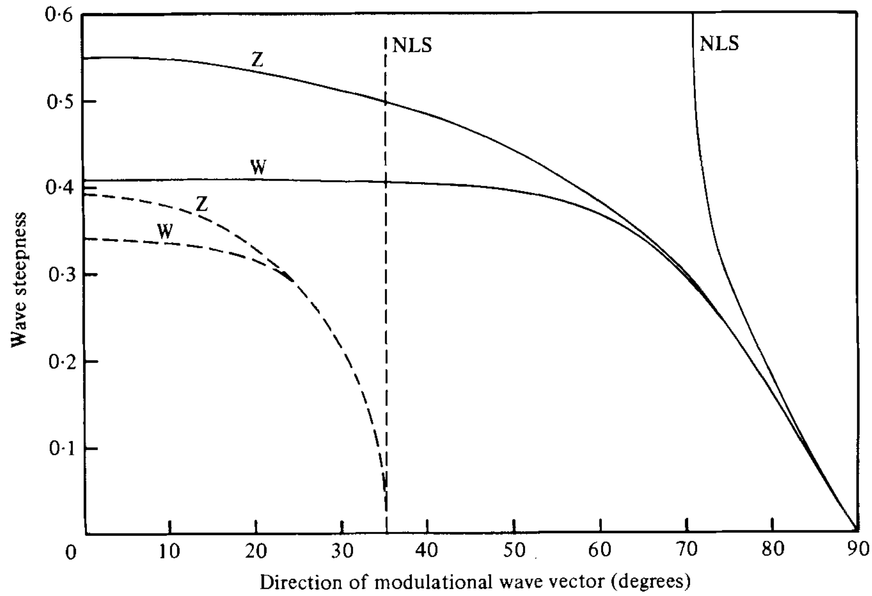


FIGURE 9. Stability boundary and neutral stability curve for infinitely long perturbations. – – –, stability boundary ($\text{Im}\,\sigma = 0$); ——, neutral stability curve ($\text{Im}\,\sigma = \text{Re}\,\sigma = 0$). NLS, result from three-dimensional nonlinear Schrödinger equation; Z, results from Zakharov equation (equation (13)); W, result from Whitham's theory (exact in this limit).

evolutions of deep-water wave trains subject to three-dimensional perturbations. This speculation has yet to be evaluated by actual computation.

In figure 9, we show the stability boundary and neutral stability curve as given by the three-dimensional nonlinear Schrödinger equation (19) and (20) and the Zakharov equation (3), in the limit $\mathbf{K} \to 0$, i.e. infinitely long modulation. In this limit, Whitham's theory (Whitham 1974, p. 505) is exact for arbitrary values of $k_0 a_0$ (see also Lighthill 1967). The governing equations for infinite wave vector modulation $\hat{\boldsymbol{\kappa}}(\mathbf{x}, t)$ and action density modulation $I(\mathbf{x}, t)$ are

$$\frac{\partial \hat{\boldsymbol{\kappa}}}{\partial t} + \left(\frac{\partial \hat{\Omega}}{\partial \mathbf{k}}\right)_0 \frac{\partial \hat{\boldsymbol{\kappa}}}{\partial \mathbf{x}} + \left(\frac{\partial \hat{\Omega}}{\partial I}\right)_0 \frac{\partial I}{\partial \mathbf{x}} = 0, \tag{22}$$

$$\frac{\partial I}{\partial t} + \left(\frac{\partial \mathbf{J}}{\partial I}\right)_0 \frac{\partial I}{\partial x} + \left(\frac{\partial \mathbf{J}}{\partial \mathbf{k}}\right)_0 \frac{\partial \hat{\boldsymbol{\kappa}}}{\partial \mathbf{x}} = 0, \tag{23}$$

where $\hat{\Omega} = \hat{\Omega}(\mathbf{k}, I)$ is the angular frequency and $\mathbf{J}(\mathbf{k}, I)$ is the action density flux of the uniform wave train of wave vector $\mathbf{k}_0$ and action density I_0, and $(\;)_0$ denote evolution at $\mathbf{k} = \mathbf{k}_0$ and $I = I_0$. These quantities are known numerically from the exact calculation for deep-water waves (see, for example, Cokelet 1977). Following Peregrine & Thomas (1979), we substitute disturbances of the form $\exp[i(\mathbf{K}.\mathbf{x} - \sigma t)]$ into (22) and (23) to get

$$\sigma = \frac{\partial \omega}{\partial k} \cos\theta + \left(\left\{\left(\frac{\partial \omega}{\partial I}\right)\left(\frac{\partial J}{\partial k} + \frac{J}{k}\tan^2\theta\right)\right\}\bigg|_{\mathbf{k}=\mathbf{K}}\right)^{\frac{1}{2}}, \tag{24}$$

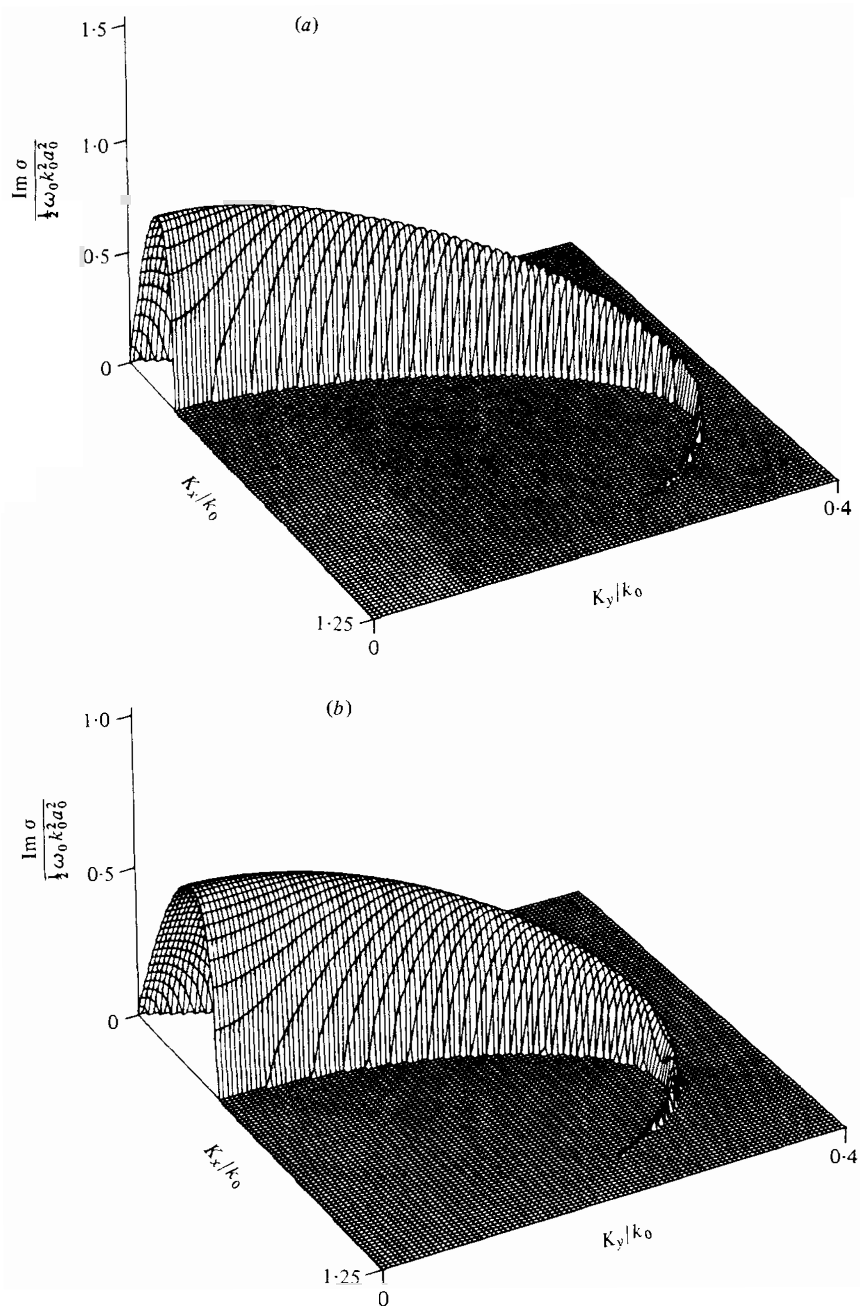


FIGURE 10. Instability growth rate for various values of wave steepness from Zakharov's equation (equation (13)). (a) $k_0 a_0 = 0{\cdot}1$, (b) $k_0 a_0 = 0{\cdot}2$, (c) $k_0 a_0 = 0{\cdot}3$, (d) $k_0 a_0 = 0{\cdot}4$.

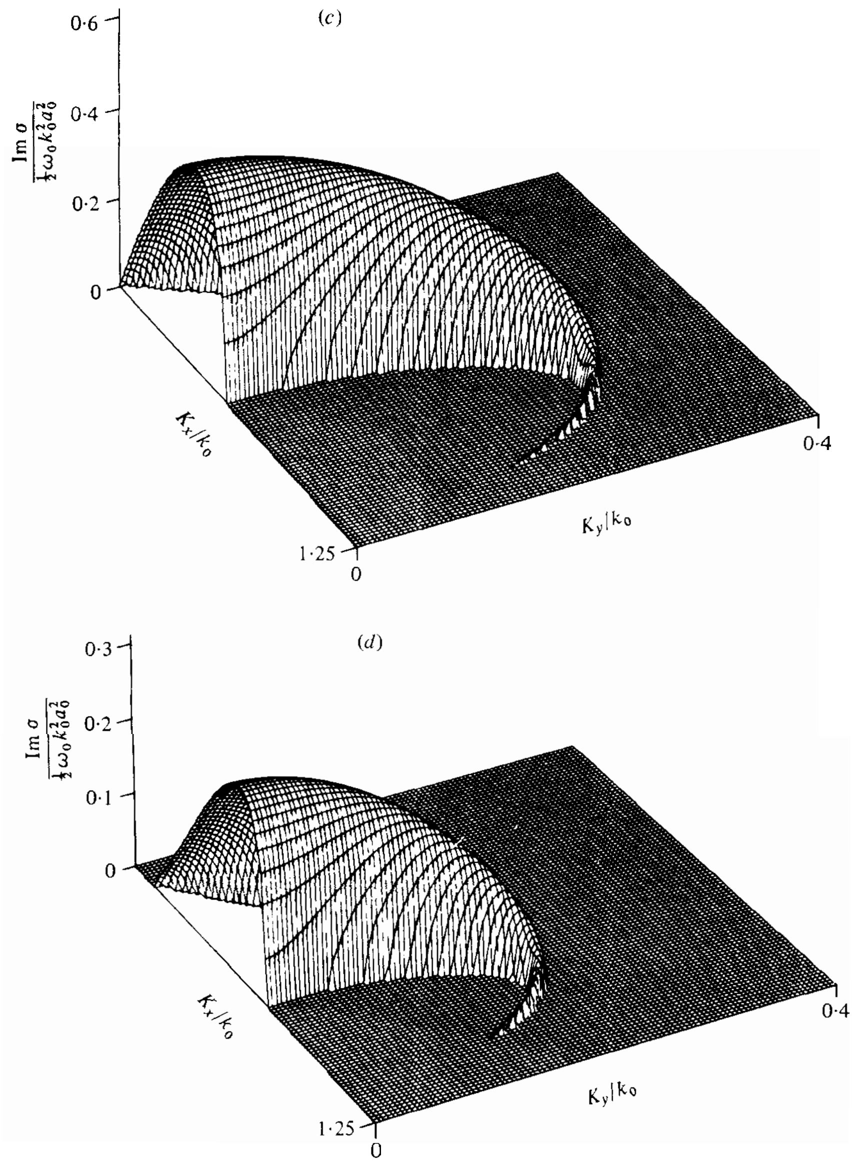


FIGURE 10 (*c*, *d*). For legend see p. 188.

where $k = |\mathbf{k}|$ and $J = |\mathbf{J}|$. The stability diagram associated with this expression is also shown in figure 9. It can be seen that for $k_0 a_0 \to 0$ (corresponding to oblique disturbances) the results of the three-dimensional nonlinear Schrödinger equation, the Zakharov equation and Whitham's theory all agree as expected. For larger values of $k_0 a_0$, the Schrödinger equation result is qualitatively incorrect. The Zakharov result, however, remains qualitatively consistent with Whitham's theory; in fact, even the quantitative agreement is surprisingly good.

Finally, we show the instability growth rate of three-dimensional perturbations for various values of $k_0 a_0$ in figure 10.

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Appendix

The interaction coefficient $T(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ was first given by Zakharov (1968) and, with some minor corrections, by Crawford *et al.* (1980). It is recorded here for reference.

Adopting the simplified notation introduced in § 2, we write

$$T(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = T_{0,1,2,3}.$$

We have

$$T_{0,1,2,3} = -\frac{2V^{(-)}_{3,3-1,1}V^{(-)}_{0,2,0-2}}{\omega_{1-3}-\omega_3+\omega_1} - \frac{2V^{(-)}_{2,0,2-0}V^{(-)}_{1,1-3,3}}{\omega_{1-3}-\omega_1+\omega_3} - \frac{2V^{(-)}_{2,2-1,1}V^{(-)}_{0,3,0-3}}{\omega_{1-2}-\omega_2+\omega_1}$$
$$-\frac{2V^{(-)}_{3,0,3-0}V^{(-)}_{1,1-2,2}}{\omega_{1-2}-\omega_1+\omega_2} - \frac{2V^{(-)}_{0+1,0,1}V^{(-)}_{2+3,2,3}}{\omega_{2+3}-\omega_2-\omega_3} - \frac{2V^{(+)}_{-2-3,2,3}V^{(+)}_{0,1,-0-1}}{\omega_{2+3}+\omega_2+\omega_3} + W_{0,1,2,3},$$

where the second-order interaction coefficients $V^{(\pm)}_{0,1,2}$ are defined as

$$V^{(\pm)}_{0,1,2} = \frac{1}{8\pi\sqrt{2}}\left\{[\mathbf{k}_0\cdot\mathbf{k}_1 \pm k_0k_1]\left[\frac{\omega_0\omega_1}{\omega_2}\frac{k_2}{k_0k_1}\right]^{\frac{1}{2}} + [\mathbf{k}_0\cdot\mathbf{k}_2 \pm k_0k_2]\left[\frac{\omega_0\omega_2}{\omega_1}\frac{k_1}{k_0k_2}\right]\right.$$
$$\left. + [\mathbf{k}_1\cdot\mathbf{k}_2 + k_1k_2]\left[\frac{\omega_1\omega_2}{\omega_0}\frac{k_0}{k_1k_2}\right]^{\frac{1}{2}}\right\}.$$

with $k_i = |\mathbf{k}_i|$, $\omega_i = \omega(k_i)$; and the third-order interaction coefficient

$$W_{0,1,2,3} = W(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$$

is defined as

$$W_{0,1,2,3} = \overline{W}_{-0,-1,2,3} + \overline{W}_{2,3,-0,-1} - \overline{W}_{2,-1,-0,3} - \overline{W}_{-0,2,-1,3} - \overline{W}_{-0,3,2,-1} - \overline{W}_{3,-1,2,-0}$$

with

$$\overline{W}_{0,1,2,3} = \frac{1}{64\pi^2}\left[\frac{\omega_0\omega_1}{\omega_2\omega_3}k_0k_1k_2k_3\right]^{\frac{1}{2}}\{2(k_0+k_1) - k_{1+3} - k_{1+2} - k_{0+3} - k_{0+2}\},$$

and

$$k_{i\pm j} = |\mathbf{k}_i \pm \mathbf{k}_j|; \quad \omega_{i\pm j} = \omega(k_{i\pm j}).$$

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